A

Abbacus School

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Abstract

The abbacus school was a school for artisans' and merchants' sons, functioning in northern Italy (thirteenth to sixteenth century). It taught the use of Hindu-Arabic numerals and fundamental commercial arithmetic: the rule of three, monetary and metrological conversions, simple and composite interest, partnership, simple and composite discounting, alloying, the technique of a "single false position," and, finally, simple area calculation. Topics like the double false position were not part of the curriculum, but they are often dealt with in the abbacus treatises; they probably served to show virtuosity in the competition for employment and pupils.

The manuscripts connected to the abbacus school tradition are of very different character. Some are messy problem collections, some orderly presentations, and a few genuine encyclopediae. Some are produced by mathematically incompetent compilers and some by the best European mathematicians of the age.

Traditionally but mistakenly, the abbacus books are to be derived from Leonardo Fibonacci's *Liber Abbaci* and *Practica Geometriae*. As shown by closer inspection of the texts, they derive from direct inspiration from a broader Mediterranean environment, which had also inspired Fibonacci around 1200. After c. 1330, however, the abbacus tradition had become an autonomous current, no more significantly influenced by the Arabic or Ibero-Provençal world. Influences from Boethian and Euclidean arithmetic, though existing, remained peripheral.

Printing allowed the preparation and spread of the great works of Luca Pacioli, Girolamo Cardano, and Niccolò Tartaglia, as well as more modest books corresponding to the school curriculum. The former group, integrated with the theory of irrationals of *Elements* X, provided the basis for the renovation of algebra brought about by Viète and Descartes; the latter made possible the spread of abbacus-type teaching of basic applied arithmetic to the whole of Western Europe, where it stayed alive until c. 1960.

Character and Beginnings

The abbacus school was a school type existing in the region between Genova, Milan, and Venice to the north and Umbria to the south from the mid-thirteenth to the sixteenth century. It was mainly frequented by merchant and artisan youth for 2 years or less around the age of 12–14 (a few younger or older), who here learned basic

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arithmetic, having already learned reading and writing and before starting apprenticeship.

Etymologically, *abbaco* obviously comes from Latin *abbacus* (< Greek αβαχ), a reckoning board. However, *abbaco* (which we may distinguish by using the most common spelling of the time) has nothing to do with reckoning boards of any kind. The first known occurrences of the word are in Leonardo Fibonacci's *Liber Abbaci* (1857) and *Practica Geometriae* (1862) and in a few references in the latter to the former as his *Liber Abbaci*. Most of these references are compatible with a generic understanding as "practical computation," but a few give more restrictive information: the *Liber abbaci* (p. 5) as well as the *Practica* (p. 8) specify that *abbacus* encompasses finger reckoning, while the *Liber abbaci* (p. 353) ascribes to the abbacus a particular way to extract the square root of a non-square, multi-digit number. It seems that Fibonacci refers to a particular teaching tradition, which could be represented by the school (or whatever it was) where his father let him pursue "the study of the abbacus" in Bejaïa for some time during his boyhood (1857, 1); but it may just as well correspond to a more widely disseminated Mediterranean type (the Italian area can probably be disregarded).

Whatever Fibonacci meant precisely by the word (if it was really his) may have been as obscure to his contemporaries as to us, but it caught on, first probably meaning "practical" or "commercial computation" and then soon referring to the new Italian school tradition that came to teach that topic. In 1241, Fibonacci received a yearly pension from the Commune of Pisa represented by its officials *in abbacatione* for his consulting services in *abbacandi* estimations (Bonaini 1857, 241) – thus probably not for teaching; but in 1265 a document refers to a master teaching the abbacus in Bologna. In the 1280s sources start talking about abbacus teachers paid by the city communes; in the longer run, only smaller cities would need to do that; Florence and Venice seem to have relied on privately run schools. Such schools remained in existence at least until the late sixteenth century – Ulivi (2002) offers a convenient survey.

Students and Teaching

The students, as mentioned, were mostly sons of artisans and merchants – including those belonging to the highest mercantile patriciate. But also Machiavelli, a lawyer's son, was sent to the abbacus school of Pier Maria Calandri (Black 2007, 379). In the city of Florence, in much of the period, between one fourth and one half of all children learned to read and write, and perhaps 10% of boys frequented the abbacus school (Høyrup 2007, 27). In other towns the percentage may have been considerably lower – not to speak of the countryside, the abbacus school was urban.

What did these boys learn? *Not* accounting – that was learned on-the-job by bank and merchant house apprentices. We happen to know two documents that outline a curriculum; one, from the earlier fifteenth century, is said to correspond to "the Pisa way" (Arrighi 1967); the other tells what was done in a Florentine school in 1519 (Goldthwaite 1972). The differences are modest and mainly concern the order in which things were taught; scattered remarks in the writings of various abbacus masters support the assumption that the two documents taken together are representative.

The initial part of the course taught operations with pure numbers. First came the writing of Hindu-Arabic numerals and, next, the multiplication tables and their application. Neither source mentions addition and subtraction – these techniques may have been implicit in the learning of the number system. Division followed, beginning with divisors known from the multiplication tables and going on with multi-digit divisors. Calculation with fractions ended the first part.

The second part of the course trained commercial mathematics (in different order in the two curricula): the rule of three, monetary and metrological conversions, simple and composite interest and reduction to interest per day, partnership, simple and composite discounting, alloying, the technique of a "single false position," and, finally, area measurement.

From the multiplication tables onward, everything was accompanied by problems to be solved as homework. More complex matters, like the use of a double false position and algebra, are amply treated in many “abbacus manuscripts” (a notion to be dealt with imminently), but they appear not to have been part of the curriculum.

The “Abbacus Books”

Though published almost four decades ago, (Van Egmond 1980) it remains an almost complete catalogue of known “abbacus manuscripts.” These have in common to have been produced by writers who appear to have been connected – as former students or as teachers – to the abbacus school environment and to deal with mathematics. Some are messy collections of problems (*zibaldoni*), while some could look like “teacher’s books” (the students themselves did not have books), presenting a sequence of topics in an orderly way (but they may also be written for patrons or friends, some indeed claim to be adequate for self-study); three are genuine mathematical encyclopediae – Benedetto da Firenze’s *Praticha d’arismetricha* from 1363 was copied often, while Ottobon. lat. 3307 from c. 1365, and the slightly earlier Florence, Bibl. Naz. Centr,. Palat. 573 are known only from their compilers’ autographs (Høyrup 2010, 32, 39). Some of the manuscripts are author’s autographs (but they may still draw heavily on earlier texts, as most mathematics textbook still do today), some are booksellers’ or similar copies, some are anonymous, some carry the name of the author, and some borrow the name of a famous author but alter or maltreat his text in one or the other way. Some authors have a deep understanding of the mathematics they present even when it is advanced, others make blunders as soon as the inverse rule of three or volumes are involved, and some cheat, either knowingly or naively plagiarizing the fraud of predecessors – the abbacus manuscript of Piero della Francesca famously falls in the latter category (Giusti 1991, 64). This mainly regards the presentation of glaringly false rules for the solution of irreducible algebraic problems of the third or fourth degree, probably used to impress mathematically incompetent municipal authorities and to dumbfound rivals in competitions for employment (Høyrup 2009).

Fibonacci Versus General Mediterranean Background

It is a recurrent claim that the abbacus books are derived from and represent reduced versions of Fibonacci’s *Liber Abbaci* and *Practica Geometriae* – representative quotations in (Høyrup 2005, 24–25). This is an almost complete mistake. The term *abbaco*, it is true, is likely to be inspired by Fibonacci, and one of the earliest known abbacus treatises also cites and draws on the *Liber abbaci*.

This treatise (Florence, Riccardiana 2404, ed. Arrighi 1989) presents itself as *Livero de l’abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa*, “Abbacus book according to the opinion of master Leonardo Fibonacci.” It is a beautiful *de luxe* vellum copy of an original which because of misinterpreted internal evidence (copied older loan documents) has been dated to 1288–1290. It must be slightly to somewhat later but hardly later than the first decade of the fourteenth century (the author does not recognize the most basic algebraic terminology). As closer analysis reveals, the text moves on two levels, clearly distinguishable in their number terminology (Høyrup 2005). On one level we find material adapted to the curriculum of the abbacus school. This level has absolutely nothing to do with the *Liber abbaci.* The other level is constituted by sophisticated problems outside the domain considered in the abbacus school. Here, almost everything is translated from the *Liber abbaci* – unfortunately, as it turns out, often without understanding. It seems that Fibonacci had already become a mythical culture hero for abbacus mathematics around 1300 – but with little real impact.

Characteristic is the way the essential rule of three is dealt with. Fibonacci does not really present a rule; he seems to describe (in terms taken from Euclidean theory) what he has observed to be done on an Arabic dust or clayboard (Arabic since inscription starts from the right, and the left is “behind”) (Fibonacci 1857, 83*f*):

> In all commercial exchanges [*negotiationes*], four proportional numbers are always found, of which three are known, but the remaining unknown. The

> first of these three known numbers is the number of sale of any merchandise, be it number, or weight, or measure [explanatory examples]. The second, however, is the price of this sale [...]. The third, then, will be the sale of some quantity of this merchandise, whose price, namely the fourth, unknown number, will not be known. Therefore, in order to find the unknown number from those that are known, we give a universal rule for all cases, namely, in the top of a board write the first number to the right, namely the merchandise. Behind in the same line you posit the price of the same merchandise, namely the second number. The third too, if it is the merchandise, write it under the merchandise, that is, under the first, and if it is the price, write it under the price, that is, under the second. In this way, as it is of the kind of that under which it is written, thus it will also be of the quality or the quantity, whether in number, in weight or in measure. [...]. When they are described thus, it will be obvious that two of those that are posited will always be contrary [*ex adverso*], which have to be multiplied together, and that if the outcome of their multiplication is divided by the third number, the fourth, unknown, will doubtlessly be found.

Riccardiana 2404 (Arrighi 1989, 9) gives the rule like this:

> If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same [kind], and divide in the other.

If one misreads Fibonacci's *ex adverso* as referring to difference in kind and not to the location in the rectangular frame which he has just described, the Riccardiana formulation may seem to abbreviate the last lines from the *Liber abbaci*. This overlooks, however, that the Riccardiana formulation had been standard in Indian and Arabic vernacular mathematics for almost a millennium (Høyrup 2012).

Another (now mutilated) early abbacus text, Siena, L. VI. 47_2, was apparently written in Pisa around 1300s; (Franci 2015) suggests a late thirteenth-century date, (Ulivi 2011), better argued, the early fourteenth century. The author might be a certain Bindo Nocchi de Ambaco, but only chronological overlap supports that assumption. A collection of numerical tables in the beginning might seem to be derived from the *Liber abbaci*. However, Fibonacci borrows a Maghreb term for prime numbers (*aṣamm*) as *hasam* and *asam*, while the Siena text uses *la salma*, clearly derived (with misunderstanding) from something like *al-aṣammāʿ*, with the article, double consonant, and a feminine ending which Fibonacci does not indicate and which must come directly from spoken Arabic. Moreover, even though the initial introduction of the rule of three has been lost (if it was ever present), it is clear from the way its use in problems is stated that it coincided with the vernacular form found in the Riccardiana manuscript.

A third early text is the "Columbia algorism" (Vogel 1977), probably the earliest of all, from 1285 to 1290 (Travaini 2003); it has been misdated to the mid-fourteenth century because a coin list was misinterpreted. It contains an explicit introduction of the rule of three – a characteristic twisted version of the vernacular rule which only turns up again in the late fifteenth century but may have survived outside the core abbacus area; however, problems calling for the use of three are mostly reduced to a counterfactual model, of the type "if 3 were 5, what would 7 be?"). This was the standard in all Ibero-Provençal abbacus-type treatises until the end of the fifteenth century and was also known to Fibonacci, who speaks of it (1857, 170) as the vernacular way – in agreement with Fibonacci's reference in one early manuscript (Vatican, Palat. lat. 1343, fol. 47^{r} col. II) to reliance on a Castilian source. There are further Iberian affinities in the Columbia algorism, so all in all it turns out that Fibonacci as well as the budding abbacus culture drew independently on both Arabic (Maghreb or al-Andalus) and Christian Ibero-Provençal sources.

This is even more clear if we look at *algebra* – a topic that did not belong to the abbacus school *curriculum* but which is dealt with in many abbacus manuscripts. As other advanced topics (in particular the double false position), it may have served in the training of assistant-apprentices – but this is a hypothesis with no support in the sources. What we do know is that proficiency in such difficult matters played a role in the competition for employment or for pupils. Algebra is dealt with in Chapter 15, Section 3 in the *Liber abbaci* – but from the very beginning, abbacus algebra is in a wholly different style, probably at first borrowed from the Ibero-Provençal area (Høyrup 2007, 147–169). One school tradition in Florence honored Fibonacci and copied a translation of his algebra, but its truly outstanding members (among whom Antonio de' Mazzinghi in the later fourteenth century and Benedetto da Firenze around 1460) made their own algebraic work in continuation of the abbacus tradition.

Autonomization

After the earlier fourteenth century, we find very few plausible instances of recent Arabic or Iberian influence; by then, the abbacus tradition had established itself as an autonomous current.

Beginning in the later fourteenth century, on the other hand, some ideas and concepts are adopted from Euclidean and Boethian mathematics into the non-curricular higher level. Antonio de Mazzinghi may have been the first to speak of the sequence of algebraic powers as a continued proportion; Benedetto da Firenze includes a chapter on the Boethian names for ratios in his encyclopedic *Praticha d'arismetricha*, while another chapter draws on Euclid and Campanus in a presentation of the composition of ratios. However, as the roughly contemporary anonymous abbacus encyclopedia Florence, Palat. 573 (fol. 17^{v}) observes concerning the Boethian names, "we in the schools do not use such terms but say instead [...] that 8 is 2/3 of 12 and 12 is 3/2 of 8." The external influence thus remained external and peripheral. This may astonish, given that several abbacus masters also taught at universities – but what they taught there was astronomy and astrology, subordinate to medicine, not mathematical theory.

Impact and Legacy

Only Luca Pacioli, Niccolò Tartaglia, and Girolamo Cardano were to achieve some measure of integration between abbacus and theoretical mathematics from the outgoing fifteenth century onward – the former two products of the abbacus environment but determined to enter the world of university learning, the latter a university scholar looking at abbacus mathematics "from a higher vantage point" (as once Fibonacci). It is no accident that all three had their works printed, but also genuine abbacus books appeared in print – first the anonymous *Larte de labbacho*, printed in Treviso (1478), and Pietro Borghi's *Opera de arithmetica* (1484); Francesco Feliciano da Lazesio's *Libro de Abbacho novamente composto* (1526) was reprinted as late as 1692.

The latter reprint is symptomatic of one of the repercussions of the abbacus tradition in later centuries. Until the new math reforms of the 1960s, the teaching of practical arithmetic in European schools (and such schools of the colonial world where European teaching was imported) remained close to the pattern established by the abbacus school. The most direct heirs were the German *Lese- und Rechenschulen*, which however also had to teach reading and writing. The spelling *Coss* of its translation of the algebraic *cosa* shows this German tradition to be derived from northern Italy (Milan etc.), which would write *cossa* – not from Tuscany or Umbria.

At the mathematically higher level, the indubitably most important legacy was *algebra* (Stedall 2010). In part mediated by Michael Stifel's *Arithmetica integra* (1544) and Simon Stevin's *L'arithmétique* (1585), the most influential works behind the renovated algebra created by Viète and Descartes were Pacioli's *Summa de arithmetica* (1494), Cardano's *Ars magna* (1545), and Rafael Bombelli's *L'algebra* (1572) – and these, on their part, owe most of their inspiration to the supracurricular work of abbacus writers, complemented by the theory of irrationals of *Elements* X.

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